

Bulk one-point function on disk in one-matrix model

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Abstract

We consider bulk correlation numbers on disk in one-matrix model. Using the recently found so-called resonance transformation from the KdV to the Liouville frame[17], we obtain an explicit expression for the bulk one-point function. The result is consistent with the form of FZZ one-point function[20] in the boundary Liouville Field Theory.

1 Introduction

Since the middle of 80's, there exist two independent approaches to 2D Quantum Gravity: the continuous one called Liouville Gravity (LG) [1, 2, 3, 4] and the discrete one using the techniques of Matrix Models (MM) [5, 6, 7, 8, 9, 10, 11]. One may refer to more references in reviews in [12, 13].

Comparison of some quantities calculated in a number of particular models using both approaches confirms the expectation about their equivalence[11, 14, 15]. The most easily checked quantities in MM are the n -point “correlation numbers”

$$C_{k_1, \dots, k_n} = \langle O_{k_1} \cdots O_{k_n} \rangle \quad (1.1)$$

where O_k is the integrated form of the local density (2-form operator) $\mathcal{O}_k(X)$

$$O_k = \int_{\mathcal{M}} \mathcal{O}_k(X) \quad (1.2)$$

over the manifold \mathcal{M} which accommodates both “matter” and the metric degrees of freedom localized at $X \in \mathcal{M}$.

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It is convenient to introduce a generating function of the correlation numbers in Liouville gravity

$$Z_{LG}(\{\lambda_k\}) = \langle e^{\sum_k \lambda_k \mathcal{O}_k} \rangle \quad (1.3)$$

which can be viewed as the partition function of the original theory perturbed by adding the fields $\mathcal{O}_k(X)$ to the action density with the coupling constant λ_k . A similar partition function, which depends on the parameters t_k , can be introduced in Matrix models.

However, a naive identification of observables and their correlation numbers obtained from two approaches does not agree in general. The reason for this difficulty lies in the ambiguity of the choice of contact terms [16, 17]. This ambiguity is equivalent to the freedom in the identification of the coupling parameters λ_k in LG with the ones t_k in MM of the same gravitational dimensions.

It was conjectured in [16] that there exists a special choice of the contact terms in MM or, equivalently, the special transformation $t_k = t_k(\{\lambda_j\})$, which ensures the coincidence of the partition functions in the Minimal Liouville gravity $MLG(2, 2p+1)$ and in the p -critical one-matrix model $OMM(p)$ for the random surfaces with the sphere topology. This relation between the parameters t_k and λ_k was obtained in [16] up to the linear terms. Its explicit form to all orders was conjectured in [17] and checked up to the 4-th order [17, 18].

The existence of a similar transformation for the general (p_1, p_2) case remains an open problem, and so much as the problem of the calculation of the correlation numbers and their comparison on the surface with non-spherical topology.

In this note, we consider the non-spherical topology, namely, the fluctuating disk and propose the explicit expression for the generating function of the n -point bulk correlation numbers on disk. In our calculation, we use the string equation [17] in the conformal frame.

Particularly, we compute the partition function and the one-point function for the one-matrix model in this topology, which were obtained earlier in [16] by means of a different way. We find our results consistent with the form of the Fateev-Zamolodchikov-Zamolodchikov one-point function [20] in the boundary Liouville Field Theory.

In section 2 we give a short summary of the known results on the $OMM(p)$. In section 3, the partition function on the random disk with a fixed length of the boundary and one-point function of the bulk operator in the $OMM(p)$ are calculated. In section 4, the corresponding answers are compared with the Liouville gravity results. Section 5 is the conclusion and discussion.

2 One-Matrix Model

It was shown in [5, 6, 7] that the scaling partition function of the $OMM(p)$ is expressed in terms of the susceptibility $u_*(t_k)$ which corresponds to the suitably chosen root of

the string equation $P(u_*) = 0$. Here, $P(u)$ is the $p + 1$ -degree polynomial of u

$$P(u, \mu, \{t_k\}) = u^{p+1} - \mu u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1} \quad (2.1)$$

where μ is the cosmological constant, and the parameters t_k describe the relevant deviations from the p -critical point.

The singular (or universal) part of the partition function is expressed[17] as

$$Z(\mu, \{t_k\}) = \frac{1}{2} \int_0^{u_*} du P^2(u, \mu, \{t_k\}). \quad (2.2)$$

It was conjectured and partly checked in [16, 17] that there exists an analytic transformation $t_k = t_k(\mu, \{\lambda_j\})$ compatible with the scale properties so that the correlation numbers, defined by expanding the partition function into series of the new variables λ_k , satisfy the same restrictions to be fulfilled in the minimal Liouville gravity. After this transformation the polynomial $P(u, t_k)$, as a function of the new parameters λ_k , takes the form (up to the factor $(p+1)!/(2p-1)!!$)

$$\mathcal{P}(u, \mu, \{\lambda_k\}) \equiv P(u, \mu, \{t_k(\lambda)\}) = u_0^{p+1} \mathcal{Q}(u/u_0, \{\lambda_k\}) \quad (2.3)$$

where

$$\mathcal{Q}(x, \{\lambda_k\}) = \sum_{n=0}^{\infty} \sum_{k_1 \dots k_n=1}^{p-1} \frac{\lambda_{k_1} \lambda_{k_2} \dots \lambda_{k_n}}{n!} \left(\frac{d}{dx}\right)^{n-1} L_{p-\sum k_i - n}(x), \quad (2.4)$$

$$\left(\frac{d}{dx}\right)^{-1} L_p(x) \equiv \int_0^x L(y) dy = \frac{L_{p+1}(x) - L_{p+1}(0)}{2p+1}. \quad (2.5)$$

$u_*(\lambda)$ is the suitably chosen root of $\mathcal{P}(u, \lambda_k)$ and reduces to u_0 if $\lambda_k = 0$ as given in Eq. (3.26) of [17]

$$u_0 = \sqrt{\frac{2(2p-1)}{p(p-1)}} \mu. \quad (2.6)$$

The partition function in the new variables is rewritten as

$$\mathcal{Z}(\mu, \{\lambda_k\}) = \frac{1}{2} \int_0^{u_*} \mathcal{Q}^2(u/u_0, \lambda_k) du \quad (2.7)$$

and is considered as the generating function of the correlation numbers in the Liouville frame.

It is important to remember that these new correlation numbers are equal to the coefficients of the expansion of the partition function $\mathcal{Z}(\mu, \{\lambda_k\})$ around the point $\lambda_k = 0$ which does not coincide with the point $t_k = 0$. The last one was used to define the correlation numbers in the KdV frame. We will refer to the new choice of the point of the expansion as the Liouville (or the conformal) background following [16].

3 Bulk one-point function on disk

Now we consider the fluctuating disk with the boundary length L . It was shown in [7, 19] that the partition function of the random disk with a finite boundary is given by

$$Z_B(\mu, L, t_1, \dots, t_{p-1}) = \frac{1}{\sqrt{L}} \int_{t_{p-1}}^{\infty} dy e^{-L u(y)} \quad (3.1)$$

where $u(y)$ stands for the solution of the string equation in the KdV frame

$$P(u, \mu, t_1, \dots, t_{p-2}, y) = u^{p+1} - \mu u^{p-1} + \sum_{k=1}^{p-2} t_k u^{p-k-1} + y = 0. \quad (3.2)$$

The disk partition function $Z_B(\mu, L, t_k = 0)$ in the KdV frame was computed by Moore, Seiberg, and Staudacher. The answer is given in Eqs. (4.4) and (4.5) of [16].

However, to compare the results of MM with the ones of MLG on disk we need to use the Liouville frame. Therefore, we will start from the formula

$$\mathcal{Z}_B(\mu, L, \{\lambda_k\}) = \frac{1}{\sqrt{L}} \int_{\lambda_{p-1}}^{\infty} dy e^{-L u(y)} \quad (3.3)$$

where $u(y)$ stands for the solution of the string equation in the Liouville frame

$$\mathcal{P}(u, \mu, \lambda_1, \dots, \lambda_{p-2}, y) = 0. \quad (3.4)$$

We can compute the partition function and the n -point functions on disk using the string equation after changing the integration variable y to u . For this we use the relation

$$\frac{dy}{du} = - \frac{d\mathcal{P}(u, \mu, \{\lambda_k\})}{du} \quad (3.5)$$

and obtain

$$\begin{aligned} \mathcal{Z}_B(\mu, L, \{\lambda_k\}) &= - \frac{u_0^{p+1}}{\sqrt{L}} \int_{x_*}^{\infty} dx \frac{dQ(x, \{\lambda_k\})}{dx} e^{-L u_0 x} \\ &= u_0^{p+2} \sqrt{L} \int_{x_*}^{\infty} dx Q(x, \{\lambda_k\}) e^{-L u_0 x} \end{aligned} \quad (3.6)$$

where the integration by parts is performed and x_* is the corresponding root of the renormalized string equation $Q(x_*, \{\lambda_k\}) = 0$. The formula (3.6) is the expression for the generating function of the correlation numbers on the disk in the Liouville frame. The disk partition function is obtained when $\lambda_k = 0$

$$\mathcal{Z}_B(\mu, L, \{\lambda_k\} = 0) = \frac{u_0^{p+1}}{\sqrt{L}} \int_1^{\infty} dx L_p(x) e^{-L u_0 x}. \quad (3.7)$$

The bulk one-point function is given as

$$\begin{aligned}\langle O_k \rangle_L &= \frac{\partial}{\partial \lambda_k} \mathcal{Z}_B(\mu, L, \lambda_1, \dots, \lambda_{p-1}) \Big|_{\{\lambda_i\}=0} \\ &= u_0^{p-k} \sqrt{L} \int_{x_*}^{\infty} dx L_{p-1-k}(x) e^{-Lu_0 x}\end{aligned}\quad (3.8)$$

One can evaluate the expressions using Eqs. (3.7) and (3.8), and the relation [17] between the Legendre polynomials and the Macdonald function of a half-integer order

$$\int_1^{\infty} dx L_n(x) e^{-px} = \sqrt{\frac{2}{\pi p}} K_{n+1/2}(p). \quad (3.9)$$

The explicit results are

$$\mathcal{Z}_B(\mu, L, \lambda_k = 0) = \sqrt{\frac{2}{\pi}} \frac{u_0^{p+1/2}}{L} K_{p+1/2}(u_0 L) \quad (3.10)$$

$$\langle O_k \rangle_L = \sqrt{\frac{2}{\pi}} u_0^{p-k-1/2} K_{p-k-1/2}(u_0 L). \quad (3.11)$$

These relations coincide with formulas (4.19) and (4.24) in [16] where they were obtained in a different way.

4 Comparison with Boundary Liouville field theory

Noting that the Liouville gravity partition function with the fixed boundary length ℓ , $Z_{BLG}(\mu, \ell, \{\lambda_k\})$ is related to the one with the fixed boundary cosmological constant μ_B , $\mathcal{Z}(\mu, \mu_B, \{\lambda_i\})$ by the inverse Laplace transform

$$Z_{BLG}(\mu, \ell, \{\lambda_k\}) = \ell \int_{\uparrow} \frac{d\mu_B}{2\pi i} e^{\mu_B \ell} \mathcal{Z}(\mu, \mu_B, \{\lambda_k\}), \quad (4.1)$$

where the contour \uparrow goes along the imaginary axis to the right from all the singularities of the integrand.

One can express the one-point correlation number with fixed ℓ in terms of the one with fixed μ_B as

$$\langle O_k \rangle_{\ell} = \ell \int_{\uparrow} \frac{d\mu_B}{2\pi i} e^{\mu_B \ell} \langle O_k \rangle_{\mu_B} \quad (4.2)$$

where

$$\langle O_k \rangle_{\mu_B} = \frac{\partial}{\partial \lambda_k} Z_{BLG}(\mu, \mu_B, \{\lambda_i\}) \Big|_{\{\lambda_i\}=0}. \quad (4.3)$$

The bulk one-point function in the Boundary Liouville field theory is obtained in [20]

$$U(\alpha|\mu_B) = \frac{2}{b} (\pi \mu \gamma(b^2))^{(Q-2\alpha)/(2b)} \Gamma(2b\alpha - b^2) \Gamma\left(\frac{2\alpha}{b} - \frac{1}{b^2}\right) \cosh((2\alpha - Q)\pi s) \quad (4.4)$$

with

$$\cosh^2(\pi bs) = \frac{\mu_B^2}{\mu} \sin(\pi b^2). \quad (4.5)$$

Using the Laplace transform, one gets

$$U(\alpha|\mu_B) = \int_0^\infty \frac{d\ell}{\ell} e^{-\mu_B \ell} W_\alpha(\ell) \quad (4.6)$$

$$W_\alpha(\ell) = \frac{2}{b} (\pi \mu \gamma(b^2))^{(Q-2\alpha)/(2b)} \frac{\Gamma(2b\alpha - b^2)}{\Gamma(1 + \frac{1}{b^2} - \frac{2\alpha}{b})} K_{(Q-2\alpha)/b}(\kappa \ell) \quad (4.7)$$

with

$$\kappa^2 = \frac{\mu}{\sin(\pi b^2)}. \quad (4.8)$$

Noting that $1/b^2 = p + 1/2$ and putting $\alpha_k = (k+2)b/2$, one has

$$\frac{Q - 2\alpha_k}{b} = p - k - \frac{1}{2} \quad (4.9)$$

which demonstrates the consistency between the matrix model and the Liouville gravity on disk since $W_{\alpha_k}(\ell)$ has the same dependence on μ and ℓ as the one in $\langle O_k \rangle_L$ in Eq. (3.11) up to the renormalization of L .

5 Conclusion and Discussion

The result (3.6) gives an expression for the generating function of the correlation numbers of the p -critical one-matrix model on disk in the Liouville frame. To confirm this proposal, we explicitly compare the one-point correlation numbers obtained from (3.6) against the FZZ one-point function Eq. (4.7) in the Boundary Liouville field theory and find the proposal consistent.

One may further evaluate the bulk n -point functions using (3.6). However, one cannot check their correctness at this moment, since the corresponding answers for MLG are not known yet.

Another interesting problem in both MM and MLG will be the computation of bulk-boundary correlation numbers and boundary correlation numbers on disk [21].

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